Example 2 We remind that

- A K3 surface is a smooth minimal algebraic surface S with $q(S)=0$ and $K_S \equiv 0$.
  K3 surfaces have invariants $K^2=0, q=0, p_g=1$, $e=24$, $\kappa(S)=0$.

- An Enriques surface is a smooth minimal algebraic surface S with $q(S)=0$ and a <u>non</u>-trivial canonical bundle for which $2K_S \equiv 0$.
  Enriques surfaces have invariants $K_S^2 = q = p_g = 0$, $e=12$, $\kappa(S)=0$

<u>Historical Remark</u>:

In 1895 Castelnuovo posed a question wether a surface with $q=p_g=0$ is necessarily rational.

Enriques answered to this question: there are examples of surfaces with $q=p_g=0$ that are not rational. These were the first examples of Enriques Surfaces.

Later on, Castelnuovo proved the beautiful

Theorem (Castelnuovo Rationality Criterion)

S is a rational surface $\iff$ $q(S)=0$ and $P_2 = h^0(2K_S) = 0$.

We observe that any Enriques Surface is the quotient of an Étale double covering of a K3 surface.

Indeed, given an Enriques Surface Y, then $2k_Y \equiv 0$, so $\mathcal{L}_{\bar{1}} := k_Y$ and $D_{\bar{1}} = 0$ defines an abelian cover $\pi: X \to Y$ with group $G = \mathbb{Z}/2$. By construction, π is étale (no ramification), so X is smooth minimal and by Riemann-Hurwitz: $K_X \equiv \pi^*(k_Y)$

Thus, $2k_X = \pi^*(2k_Y) = 0$, and since $\mathcal{L}_{\bar{1}} = k_Y$, then
$$p_g(X) = h^0(X, K_X) = h^2(X, \mathcal{O}_X) = h^2(Y, \mathcal{O}_Y) + h^2(Y, \mathcal{L}_{\bar{1}}^{-1}) \underset{\text{Serre Duality}}{=} h^0(Y, k_Y) + h^0(Y, \underbrace{k_Y + \mathcal{L}_{\bar{1}}}_{k_Y + \mathcal{L}_{\bar{1}}}) = 0 + 1 = 1$$

So $2k_X \equiv 0$, $p_g(X) = 1 \Rightarrow k_X \equiv 0$.

Instead, $q(X) = h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y) + h^1(Y, \underbrace{\mathcal{L}_{\bar{1}}^{-1}}_{-k_Y = k_Y})$
$$= 2h^1(Y, \mathcal{O}_Y) = 2q(Y) = 0$$

Thus, X is a K3 surface.

<u>Remark</u> K3 surfaces are always simply connected, so the universal cover of an Enriques surface is always a K3 surface.

<u>WARNING</u>: The opposite is false, namely is <u>not</u>

true that any K3-surface has an involution $\sigma \in \mathrm{Aut}(K3)$ s.t $K3/\langle\sigma\rangle$ is Enriques.

Lemma 1 (the divisors D_g determine $\mathcal{L}_\chi$ up to torsion).
(F.-Pignatelli 2023, Cor. 1.6, Pard., Prop. 2.1)

Any invertible sheaf $\mathcal{L}_\chi$, $\chi \in G^*$, satisfies

$$|\chi| \cdot \mathcal{L}_\chi \equiv \sum_{g\in G} \frac{|\chi| \cdot r_\chi^g}{|g|} D_g$$

it denotes the order of χ

proof We prove by Induction on k that

$$k \cdot \mathcal{L}_\chi \equiv \mathcal{L}_{\chi^k} + \sum_{g\in G} \left\lfloor \frac{k r_\chi^g}{|g|} \right\rfloor D_g$$

$k=1$ trivial, so suppose the thesis holds for $k-1$.

We use Pardini Equation: $\mathcal{L}_\chi + \mathcal{L}_{\chi^{k-1}} \equiv \mathcal{L}_{\chi^k} + \sum_{r_\chi^g + r_{\chi^{k-1}}^g \geq |g|} D_g$

However, by inductive hypothesis, we have:

$$\mathcal{L}_\chi + (k-1)\mathcal{L}_\chi - \sum_{g\in G} \left\lfloor \frac{(k-1) r_\chi^g}{|g|} \right\rfloor D_g \equiv \mathcal{L}_{\chi^k} + \sum_{r_\chi^g + r_{\chi^{k-1}}^g \geq |g|} D_g$$

$$\Rightarrow k\mathcal{L}_\chi \equiv \mathcal{L}_{\chi^k} + \sum_{g\in G} \left\lfloor \frac{(k-1) r_\chi^g}{|g|} \right\rfloor D_g + \sum_{r_\chi^g + r_{\chi^{k-1}}^g \geq |g|} D_g \quad (*)$$

Fixed $g\in G$, $(k-1) r_\chi^g = \alpha |g| + r_{\chi^{k-1}}^g$, where $\alpha = \left\lfloor \frac{(k-1) r_\chi^g}{|g|} \right\rfloor$ by definition.

Then
$$r_\chi^g + r_{\chi^{k-1}}^g = r_\chi^g + (k-1) r_\chi^g - \alpha|g| = k r_\chi^g - \alpha |g| \geq |g| \Leftrightarrow$$
$$k r_\chi^g \geq (\alpha+1)|g| \Leftrightarrow \left\lfloor \frac{k r_\chi^g}{|g|} \right\rfloor = \alpha + 1$$

Instead, if $k r_\chi^g < (\alpha+1)|g| \Rightarrow \left\lfloor \frac{k r_\chi^g}{|g|} \right\rfloor = \alpha$

In both cases, D_g occurs in $(*)$ with multiplicity $\left\lfloor \frac{k r_\chi^g}{|g|} \right\rfloor$.

Finally, for $k = |\chi|$, then $L_{\chi^k} = L_{\mathbb{1}} = \mathcal{O}$, and $\frac{|\chi| r_\chi^g}{|g|}$ is an integer because $\chi(g) = e^{\frac{2\pi i}{|g|} r_\chi^g}$

$$\Rightarrow \chi^{|\chi|}(g) = 1 = e^{2\pi i \cdot \frac{|\chi| r_\chi^g}{|g|}}$$ $\blacksquare$

As we have seen in Example 1 above, verifying that a set of building data determine an abelian cover can become computational challenging as the number of equations to satisfy grows quadratically with the order of G. The next proposition reduces the number of linear equations to satisfy to just the lenght of the group G.

Def (reduced building data) Let $\pi: X \to Y$ be a G-cover, Y smooth, X normal, with build. data $\{L_\chi\}_\chi, \{D_g\}$. Assume $G^* = \bigoplus_{i=1}^{s} \langle \chi_i \rangle$. Then $\{L_{\chi_i}\}_{i=1,\dots,s}, \{D_g\}_{g\in G}$ is called a set of reduced building data of π.

Proposition (Pardini, Prop. 2.1)

Let Y be a smooth algebraic variety and let G be an abelian group.

Let $\chi_1, \dots \chi_s$ be some generators of G^*, so

$$G^* = \langle \chi_1, \dots \chi_s \rangle .$$

Given an abelian cover $\pi : X \to Y$ with group G and building data $\{L_\chi\}_{\chi \in G^*}$, $\{D_g\}_{g \in G}$, then

$$\forall i = 1, \dots s, \qquad |\chi_i| \, L_{\chi_i} \equiv \sum_{g \in G} \frac{|\chi_i| \, r_{\chi_i}^g}{|g|} D_g . \qquad (\star)$$

Conversely, given a collection $\{L_{\chi_1}, \dots, L_{\chi_s}\}$ of line bundles of Y and divisors $\{D_g\}_{g \in G}$ such that $(\star)$ hold for any $\chi_1, \dots \chi_s$, then it there exists a abelian cover $\pi : X \to Y$ with group G.

When X is normal, then π has reduced building data $\{L_{\chi_1}, \dots L_{\chi_s}\}$, $\{D_g\}_{g \in G}$.

If Y is complete, then the reduced building data determine the cover up to isomorph. of Galois covers.

proof Given an abelian cover π, then $(\star)$ hold by the previous Lemma 1.

Furthermore, $\{L_\chi\}_{\chi \in G^*}$ are determined by $\{L_{\chi_1}, \dots L_{\chi_s}\}$ and $\{D_g\}_{g \in G}$. To see this, we remind that:

1) $r^{g}_{\eta^s}$ is the reminder of $s \cdot r^{g}_{\eta}$ by the division with $|g|$

2) $\left\lfloor \frac{a+b}{c} \right\rfloor = \left\lfloor \frac{a}{c} \right\rfloor + \left\lfloor \frac{b}{c} \right\rfloor + \left\lfloor \frac{r_1+r_2}{c} \right\rfloor$, where $a = \alpha_1 c + r_1$, $b = \alpha_2 c + r_2$

Thus, let us consider $x = x^{a_1} \cdots x_s^{a_s} \in G^*$.

We need to write L_x in fuction of $\{\{L_{x_i}\}_{i=1}^{s}, \{D_g\}_g\}$:

$$L_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + L_{x_s^{a_s}} = L_x + \sum_{g \in G} \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + r^{g}_{x_s^{a_s}}}{|g|} \right\rfloor D_g$$

$$\Rightarrow L_x \underset{\text{by Lemma 1}}{=} L_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + a_s \cdot L_{x_s} - \sum_{g \in G} \left\lfloor \frac{a_s r^{g}_{x_s}}{|g|} \right\rfloor D_g - \sum_{g \in G} \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + r^{g}_{x_s^{a_s}}}{|g|} \right\rfloor D_g$$

$$= L_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + a_s L_{x_s} - \sum_{g \in G} \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + a_s r^{g}_{x_s}}{|g|} \right\rfloor D_g$$

$a_s r^{g}_{x_s} = \alpha |g| + r^{g}_{x_s^{a_s}}$

and by 2) we have

$$\left\lfloor \frac{\overset{a}{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}}} + \overset{b}{a_s r^{g}_{x_s}}}{|g|} \right\rfloor = 0 + \left\lfloor \frac{a_s r^{g}_{x_s}}{|g|} \right\rfloor + \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + r^{g}_{x_s^{a_s}}}{|g|} \right\rfloor$$

This computation suggests that

$$L_x = \sum_{i=1}^{s} a_i L_{x_i} - \sum_{g \in G} q^{g}_{x} D_g, \text{ where } q^{g}_{x} := \left\lfloor \frac{\sum_{i=1}^{s} a_i r^{g}_{x_i}}{|g|} \right\rfloor$$

Let us prove it by induction on s:

$s = 1$: This is Lemma 1.

By the previous computation, we have

$$L_x = L_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + a_s L_{x_s} - \sum_{g \in G} \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + a_s r^{g}_{x_s}}{|g|} \right\rfloor D_g$$

$$\underset{\text{inductive hypothesis}}{=} \sum_{i=1}^{s} a_i L_{x_i} - \sum_{g \in G} \Bigg(\underbrace{\left\lfloor \frac{\sum_{i=1}^{s-1} a_i r^{g}_{x_i}}{|g|} \right\rfloor + \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + a_s r^{g}_{x_s}}{|g|} \right\rfloor} \Bigg) D_g$$

$$\left\lfloor \frac{\sum_{i=1}^{s} a_i r^{g}_{x_i}}{|g|} \right\rfloor \underset{2)}{=} \left\lfloor \frac{\sum_{i=1}^{s-1} a_i r^{g}_{x_i}}{|g|} \right\rfloor + \left\lfloor \frac{a_s r^{g}_{x_s}}{|g|} \right\rfloor + \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + r^{g}_{x_s^{a_s}}}{|g|} \right\rfloor \overset{2)}{=} \left\lfloor \frac{\sum_{i=1}^{s-1} a_i r^{g}_{x_i}}{|g|} \right\rfloor + \left\lfloor \frac{r^{g}_{x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}} + a_s r^{g}_{x_s}}{|g|} \right\rfloor$$

$$= \sum_{i=1}^{s} a_i L_{x_i} - \sum_{g\in G} q_x^g D_g$$ and the thesis follows.

Conversely, let us consider a set of data $\{L_{x_i}\}_{i=1}^{s}$, $\{D_g\}_g$ of Y satisfying (✪). To prove the existence of a covering π, It is sufficient to define the remain L_x, $x\in G^*$, such that $\{L_x\}_{x\in G^*}$ and $\{D_g\}_g$ satisfy Pardini Equations.

The previous computation suggests how to define L_x:

$$\boxed{L_x := \sum_{i=1}^{s} a_i L_{x_i} - \sum_{g\in G} q_x^g D_g}, \text{ where } q_x^g := \left\lfloor \frac{\sum_{i=1}^{s} a_i r_{x_i}^g}{|g|} \right\rfloor$$

Let us prove $\{L_x\}_{x\in G^*}$ and $\{D_g\}_{g\in G}$ satisfy Pardini Equations:

$$L_x + L_\eta \underset{\substack{\downarrow \\ x = x_1^{a_1}\cdots x_s^{a_s} \\ \eta = x_1^{b_1}\cdots x_s^{b_s}}}{=} \sum_{i=1}^{s} a_i L_{x_i} - \sum_{g\in G} q_x^g D_g + \sum_{i=1}^{s} b_i L_{x_i} - \sum_{g\in G} q_\eta^g D_g$$

$$= \sum_{i=1}^{s} (a_i + b_i) L_{x_i} - \sum_{g\in G} (q_x^g + q_\eta^g) D_g$$

$$= L_{x\eta} + \sum_{g\in G} \left\lfloor \frac{r_x^g + r_\eta^g}{|g|} \right\rfloor D_g$$ and we have done.

if we prove $q_{x\eta}^g = q_x^g + q_\eta^g - \left\lfloor \frac{r_x^g + r_\eta^g}{|g|} \right\rfloor$

$$q_x^g + q_\eta^g = \left\lfloor \frac{\sum_{i=1}^{s} a_i r_{x_i}^g}{|g|} \right\rfloor + \left\lfloor \frac{\sum_{i=1}^{s} b_i r_{x_i}^g}{|g|} \right\rfloor \underset{\substack{\downarrow \\ \text{by 2)}}}{=} \left\lfloor \frac{\sum a_i r_{x_i}^g}{|g|} \right\rfloor + \left\lfloor \frac{\sum b_i r_{x_i}^g}{|g|} \right\rfloor - \left\lfloor \frac{r_x^g + r_\eta^g}{|g|} \right\rfloor$$

$\blacksquare$

Example 3 If Y is simply connected, then $\mathrm{Pic}(Y)$ has not torsion, i.e.

$$\forall D \in \mathrm{Pic}(Y) \qquad nD = 0 \implies D = 0.$$

Indeed, if $nD = 0$, then we can choose $G = \mathbb{Z}/n$ and the reduced building data $\mathcal{L}_{\chi} := D$, $D_g = 0 \ \forall g \in G$. Pardini Equation becomes $nD \equiv 0$, which is satisfied by assumption, so this would define an étale $\mathbb{Z}_n$ - covering $\pi : X \to Y$ with degree $\deg(\pi) = n$. Thus, π is a topological covering to a simply connect. top. space $\Rightarrow \pi$ is an iso, namely $n = \deg(\pi) = 1$.

Let us consider an abelian cover $\pi : X \to Y$ with reduced building data $\{\mathcal{L}_{\chi_1}, \dots \mathcal{L}_{\chi_s}\}, \{D_g\}_{g \in G}$. Then by Lemma 1: $|\chi_i| \mathcal{L}_{\chi_i} \equiv \sum_{g \in G} \frac{|\chi|r_\chi^g}{|g|} D_g$, so $\mathcal{L}_{\chi_i}$ is determined by D_g up to torsion.

Thus, if $\mathrm{Pic}(Y)$ has not torsion (such as for rational varieties ($\mathbb{P}^n$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_m$), which are simply connected) then $\mathcal{L}_{\chi_i}$ are uniquely determined by the D_g.

The only condition to ensure that $\{D_g\}_{g \in G}$ define an abelian covering with group G is that

$\sum_{g\in G} \frac{|\chi_i| r_{\chi_i}^g}{|g|} D_g$ is $|\chi_i|$-divisible (namely it is linearly equivalent to $|\chi|$-times another divisor).

In this case, we would define

$$L_{\chi_i} := \frac{1}{|\chi_i|} \cdot \sum_{g\in G} \frac{|\chi_i| r_{\chi_i}^g}{|g|} D_g$$

Then, by the previous Prop, $\{L_{\chi_1}, \dots L_{\chi_s}\}, \{D_g\}_g$ are reduced building data of a unique ab. covering, whose remaining L_χ are given by:

$$L_\chi := \frac{1}{|\chi|} \sum_{g\in G} \frac{|\chi| r_\chi^g}{|g|} D_g, \text{ by Lemma 1.}$$

Def

For this reason, when $Pic(Y)$ has not torsion, then we say that a collection $\{D_g\}_{g\in G}$ of Y, for which $\sum_{g\in G} \frac{|\chi_i| r_{\chi_i}^g}{|g|} D_g$ is $|\chi_i|$-divisible, defines a unique abelian cover $\pi: X \to Y$ with group G.

Example 4 Let $G = \overset{\langle e_1, e_2\rangle}{\overset{\shortparallel}{\mathbb{Z}/_5 \times \mathbb{Z}/_5}}$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$.

Consider $D_{e_1} := 0 \times \mathbb{P}^1$, $D_{e_2} := 1 \times \mathbb{P}^1$, $D_{4e_1+4e_2} := \infty \times \mathbb{P}^1$

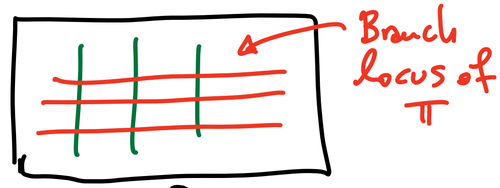
$D_{e_1+2e_2} := \mathbb{P}^1 \times 0$, $D_{3e_1+4e_2} := \mathbb{P}^1 \times 1$, $D_{e_1+4e_2} := \mathbb{P}^1 \times \infty$

$D_g = 0$ otherwise

$Pic(Y) = \langle H_1, H_2 \rangle_{\mathbb{Z}}$, so let us verify that, given $\varepsilon_1, \varepsilon_2$ dual basis of e_1, e_2 on $G^* = \langle \varepsilon_1, \varepsilon_2 \rangle$, $\sum_{g\in G} \frac{|\varepsilon_i| r_{\varepsilon_i}^g}{|g|} D_g$ is

$|\varepsilon_i|$ - divisible:
$\overset{\shortparallel}{5}$

$\mathbb{P}^1 \times \mathbb{P}^1$



$$\sum_{g\in G} \frac{|\varepsilon_1| r_{\varepsilon_1}^g}{|g|} D_g = \sum_{g\in G}' r_{\varepsilon_1}^g D_g = D_{e_1} + 4D_{4e_1+4e_2} + D_{e_1+2e_2} + 3D_{3e_1+4e_2} + D_{e_1+4e_2}.$$

$$\equiv H_1 + 4H_1 + H_2 + 3H_2 + H_2$$

$$\equiv 5(H_1+H_2) \quad \Rightarrow \mathcal{L}_{\varepsilon_1} := \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(H_1+H_2)$$

$$\sum_{g\in G} \frac{|\varepsilon_2| r_{\varepsilon_2}^g}{|g|} D_g = \sum_{g\in G}' r_{\varepsilon_2}^g D_g = D_{e_2} + 4D_{4e_1+4e_2} + 2D_{e_1+2e_2} + 4D_{3e_1+4e_2} + 4D_{e_1+4e_2}.$$

$$\equiv H_1 + 4H_1 + 2H_2 + 4H_2 + 4H_2$$

$$\equiv 5(H_1+2H_2) \quad \Rightarrow \mathcal{L}_{\varepsilon_2} := \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(H_1+2H_2)$$

Thus $\{D_g\}_{g\in(\mathbb{Z}/5)^2}$ in $\mathbb{P}^1\times\mathbb{P}^1$ defines a unique abelian covering $\pi: S \to \mathbb{P}^1\times\mathbb{P}^1$ with group $G=(\mathbb{Z}/5)^2$.

Def S is called Beauville surface. It is a smooth surface of general type (Kodaira dim. 2) with $K_S^2 = 8$, $p_g(S) = q(S) = 0$, $\chi(S) = 1$, $e(S) = 4$.

It has been constructed by Beauville in his famous book in an alternative way than how we did (see Exercise X.13(4) in Beauville, Complex Algebraic Surfaces).

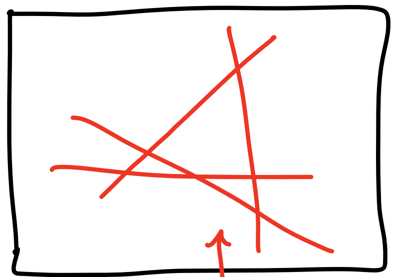
It is a very important example in the literature of algebraic surfaces of general type.

Example 5 Let $G = (\mathbb{Z}/n)^k = \langle e_1, \dots, e_n \rangle$, $Y = \mathbb{P}^2$, and

let $D_{e_1} := l_1, \dots\dots D_{e_k} := l_n$, $D_{(n-1)\sum_{i=1}^{k} e_i} := l_{k+1}$, $D_g = 0$ otherwise

be $(k+1)$-lines.

$\mathbb{P}^2$



Let $G^* = \langle \varepsilon_1, \dots \varepsilon_n \rangle$ with ε_i dual character of e_i: $\varepsilon_i(e_j) = e^{\frac{2\pi i}{n} \cdot \delta_{ij}}$.

$$\sum_{g \in G} \frac{|\varepsilon_j| \, r^{g}_{\varepsilon_j}}{|g|} D_g = \sum_{i=1}^{n} \frac{|\varepsilon_j| \, r^{e_i}_{\varepsilon_j}}{|e_i|} D_{e_i} + \frac{|\varepsilon_j| \, r^{(n-1)\sum e_i}_{\varepsilon_j}}{|(n-1)\sum e_i|} \cdot D_{(n-1)\sum e_i}$$

branch locus of π.

$$= D_{e_1} + 0 + \dots + 0 + (n-1) D_{(n-1)\sum e_i} \equiv nH$$

$$\Rightarrow \mathcal{L}_{\varepsilon_j} := \mathcal{O}_{\mathbb{P}^2}(H) \quad \forall j = 1, \dots n$$

Thus $\{D_g\}_{g \in G}$ defines an abelian covering $\pi : X \to \mathbb{P}^2$ with group $G = (\mathbb{Z}/n)^k$.

Def $\pi : X \to \mathbb{P}^2$ is called abelian cover of Hirzebruch Type. Hirzebruch studied and classified these kind of coverings in "Arrangements of lines and Algebraic Surfaces" (in the sense that he established for any possible configuration of lines what type of surfaces are in Enriques-Kodaira classification). Most of them are smooth, but if we choose special configuration of lines, then they become singular. He studied the slope

$K_S^2/\chi(S)$ of the minimal model S of X and found 3 beautiful examples satisfying Bogomolov - Miyaoka - Yau equality

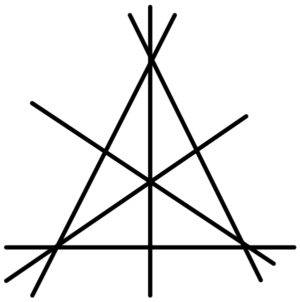
$$K_S^2 = 9\chi(\mathcal{O}_S).$$

It is called BMY-Inequality

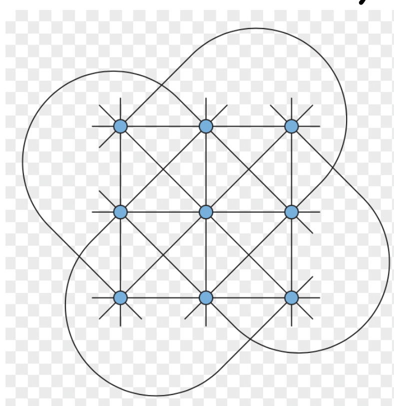
(They proved $\forall S$ of gen. type $K_S^2 \leq 9\chi(\mathcal{O}_S)$, Fields Medal 1982)

These examples are obtained with the following configuration of lines:

$G = \mathbb{Z}_5^5$



$G = \mathbb{Z}_3^{11}$ (Hesse Pencil)



$G = \mathbb{Z}_5^8$